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FUNDAMENTAL SOLUTIONS AND BOUNDARY INTEGRAL EQUATIONS FOR REISSNER'S PLATES ON TWO PARAMETER FOUNDATIONS

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Abstract—In this paper the fundamental solutions and boundary integral equations for Reissner's plates on a two parameter foundation are presented. The construction of the fundamental solutions and the numerical treatment of the boundary integral equations are described in detail. Some numerical examples are studied to demonstrate the correctness and accuracy of the formulation presented.

INTRODUCTION

The flexural behaviour of plates on elastic foundations is of interest for the design of many engineering problems. In general, the analysis of this problem is based on the incorporation of the foundation reaction into the corresponding differential equation of plates. In an attempt to find a physically close and mathematically simple representation for this problem, different models of elastic foundations have been presented. The simplest model is the Winkler model which is called a one parameter model. The main disadvantage of this model is the discontinuity of the displacements on the boundary of plates. To overcome this difficulty, two parameter models have been developed. A review of two parameter models as well as the proper mathematical formulation of them has been made by Kerr (1964) and Selvadurai (1979).

In recent years, the boundary element method has been widely used for solving the bending problems of plates. There are a few papers concerning the application of the boundary element method to the bending problems of thin plates on the Winkler foundation. Vander Weeën (1982) first presented a boundary integral equation formulation for Reissner's plate. However, to the authors' knowledge, there are no results by BEM for Reissner plates on two parameter foundations.

In this paper the fundamental solutions for Reissner's plates on a two parameter foundation are presented by the Hörmander method (1976) and the introduction of auxiliary functions. A boundary integral equation formulation for Reissner's plates on a two parameter foundation is established by the method of weighted residuals. Some numerical examples are studied and compared with analytical solutions for different boundary conditions. Numerical results show that the present method has quite good accuracy and high efficiency.

BASIC EQUATIONS

In what follows, the governing equations for Reissner's plates on a two parameter foundation are reviewed. Throughout this paper, repeated indices imply the summation convention of Einstein. Greek indices take values in the range $\{1, 2\}$, and Roman indices in the range $\{1, 2, 3\}$. The plane χ_α is assumed to coincide with the midsurface of the plate whose constant thickness is h. The distributed transverse load in the χ_3-direction is q. ψ_α denotes the rotations in the χ_α-direction and W the deflection in the thickness direction.

Relations between the generalized displacements (ψ_α and W) and forces ($M_{\alpha\beta}$ and Q_α) can be written as (Vander Weeën, 1982):

$$M_{\alpha\beta} = D\frac{1-\mu}{2}\left(\psi_{\alpha,\beta}+\psi_{\beta,\alpha}+\frac{2\mu}{1-\mu}\psi_{\gamma,\gamma}\delta_{\alpha\beta}\right)$$

$$Q_\alpha = D\frac{1-\mu}{2}\lambda^2(\psi_\alpha+W_{,\alpha}), \tag{1}$$

where $D = Eh^3/12(1-\mu^2)$ is the flexural rigidity of the plate, E is Young's modulus, μ is Poisson's ratio, and $\lambda = \sqrt{10}/h$ is a characteristic quantity of Reissner's model.

The equilibrium equations of the plate are given by:

$$M_{\alpha\beta,\beta}-Q_\alpha = 0$$

$$Q_{\alpha,\alpha}+q-p = 0, \tag{2}$$

in which p is the interface pressure of the plate and foundation. p can be expressed as (Kerr, 1964):

$$p = k_f W - G_f\nabla^2 W, \tag{3}$$

where k_f and G_f are the two parameters characterizing foundation material. For the Pasternak type foundation (Kerr, 1964), k_f and G_f are the subgrade reaction coefficient and the shear modulus of the foundation, respectively. ∇^2 is the Laplace operator.

The generalized tractions and displacements on the boundary are

$$P_\alpha = M_{\alpha\beta}n_\beta, \quad P_3 = Q_\alpha n_\alpha, \quad M_n = M_{\alpha\beta}n_\alpha n_\beta$$

$$M_{nt} = M_{\alpha\beta}t_\alpha n_\beta, \quad \psi_n = \psi_\alpha n_\alpha, \quad \psi_t = \psi_\alpha t_\alpha, \tag{4}$$

in which n_α and t_α denote the direction cosines of the outward normal and tangent to the boundary S of the plate, respectively. Appropriate boundary conditions can be expressed in the form:

(1) Simply-supported boundary

$$\psi_t = 0, \quad W = 0, \quad M_n = 0. \tag{5a}$$

(2) Clamped boundary

$$\psi_t = 0, \quad \psi_n = 0, \quad W = 0. \tag{5b}$$

(3) Free boundary.

In this case there are two regions which interact at the boundary S. The plate region Ω_p with deflection W_p is governed by eqns (2), and the surrounding foundation region Ω_f with deflection W_f is described by the second order equation for the Pasternak type foundation, which can be written as:

$$G_f\nabla^2 W_f(x) - k_f W_f(x) = 0, \quad \text{in} \quad \Omega_f. \tag{6}$$

As was shown by Kerr (1964) there are four boundary conditions on S:

$$W_p = W_f,\quad M_n = 0,\quad M_{nt} = 0,\quad P_3 + G_f\frac{\partial W_p}{\partial n} = G_f\frac{\partial W_f}{\partial n},\quad \text{on}\quad S. \tag{7}$$

By substituting eqn (1) into eqn (2), eqn (2) can be written as follows:

$$\Delta^*_{ij}U_j + b_i = 0, \tag{8}$$

where U_j denotes ψ_α and W, respectively; b_i is 0, 0, and q respectively; Δ^*_{ij} is the differential operator, which is:

$$\Delta^*_{\alpha\beta} = \left(D\frac{1-\mu}{2}\nabla^2 - C\right)\delta_{\alpha\beta} + D\frac{1+\mu}{2}\frac{\partial^2}{\partial\chi_\alpha\,\partial\chi_\beta}$$

$$\Delta^*_{\alpha 3} = -\Delta^*_{3\alpha} = -C\frac{\partial}{\partial\chi_\alpha}$$

$$\Delta^*_{33} = (C+G_f)\nabla^2 - k_f, \tag{9}$$

where $C = D\left(\frac{1-\mu}{2}\right)\lambda^2$.

FUNDAMENTAL SOLUTIONS

The fundamental solutions play an important role in the derivation of the boundary integral equations. In this section, the construction of the fundamental solutions for Reissner's plates on a two parameter foundation will be discussed in detail. The differential equation for finding the fundamental solutions is:

$$\Delta^*_{ij}U^*_{kj}(\zeta, x) = -\delta(\zeta, x)\delta_{ki}, \tag{10}$$

in which $\delta(\zeta, x)$ is the Dirac delta function, ζ is the source point and x is a field point (Vander Weeën, 1982). Equation (10) is used to represent an infinite plate under the action of a unit point at point ζ in the direction k. Following Hörmander's method (1976), the solutions of eqn (10) can be written in the following form:

$$U^*_{kj}(\zeta, x) = {}^{co}\Delta^*_{jk}\phi(\zeta, x), \tag{11}$$

where $\phi(\zeta, x)$ is an unknown scalar function, ${}^{co}\Delta^*$ is the cofactor matrix of Δ^*, which is:

$${}^{co}\Delta^*_{\alpha\beta} = [D(C+G_f)\nabla^4 - (Dk_f + CG_f)\nabla^2 + Ck_f]\delta_{\alpha\beta}$$
$$- \frac{\partial^2}{\partial\chi_\alpha\,\partial\chi_\beta}\left[D(C+G_f)\frac{1+\mu}{2}\nabla^2 + C^2 - Dk_f\frac{1+\mu}{2}\right]$$

$${}^{co}\Delta^*_{\alpha 3} = -{}^{co}\Delta^*_{3\alpha} = C\frac{\partial}{\partial\chi_\alpha}\left(D\frac{1-\mu}{2}\nabla^2 - C\right)$$

$${}^{co}\Delta^*_{33} = (D\nabla^2 - C)\left(D\frac{1-\mu}{2}\nabla^2 - C\right). \tag{12}$$

Substituting eqn (11) into eqn (10) and defining two coefficients $\alpha = (Dk_f + CG_f)/[D(C+G_f)]$ and $\beta = Ck_f/[D(C+G_f)]$, we obtain

$$D^3\left(\frac{1-\mu}{2}\right)^2\lambda^2\left(1+\frac{G_f}{C}\right)(\nabla^4 - \alpha\nabla^2 + \beta)(\nabla^2 - \lambda^2)\phi(\zeta, x) = -\delta(\zeta, x). \tag{13}$$

Equation (13) is a sixth order differential equation with a scalar function as the unknown. In order to reduce eqn (13), we introduce the two auxiliary functions which are the fundamental solutions of second and fourth order differential equations, respectively. In the above equation, we assume:

$$(\nabla^2-\lambda^2)\phi(\zeta,x) = A(\zeta,x) \tag{14a}$$

$$(\nabla^4-\alpha\nabla^2+\beta)\phi(\zeta,x) = B(\zeta,x). \tag{14b}$$

Equation (13) can be written as the following two equations:

$$D^3\left(\frac{1-\mu}{2}\right)^2\left(1+\frac{G_f}{C}\right)(\nabla^2-d^2)(\nabla^2-e^2)A(\zeta,x) = -\delta(\zeta,x) \tag{15a}$$

$$D^3\left(\frac{1-\mu}{2}\right)^2\left(1+\frac{G_f}{C}\right)(\nabla^2-\lambda^2)B(\zeta,x) = -\delta(\zeta,x), \tag{15b}$$

in which $d^2 = [\kappa+(\kappa^2-1)^{1/2}]/l^2$, $e^2 = [\kappa-(\kappa^2-1)^{1/2}]/l^2$, $l^4 = 1/\beta$ and $\kappa = \alpha l^2/2$.

With the aid of eqns (14), $\phi(\zeta,x)$ can be expressed with a linear combination of the auxiliary functions and their derivatives as follows:

$$\phi(\zeta,x) = (\lambda^4-\alpha\lambda^2+\beta)^{-1}((\alpha-\lambda^2)A(\zeta,x)+B(\zeta,x)-\nabla^2A(\zeta,x)). \tag{16}$$

From the expressions for d and e, we know that the fundamental solution of eqn (15a) consists of three cases, i.e. $\kappa = 1$, $\kappa > 1$ and $0 \leqslant \kappa < 1$. According to the Fourier transform, the solutions of eqns (15) are:

$$A(\zeta,x) = -\frac{l^2}{\pi\sigma\sqrt{\kappa^2-1}}(K_0(er)-K_0(\mathrm{d}r)), \quad \kappa > 1 \tag{17a}$$

$$A(\zeta,x) = -\frac{l}{\pi\sigma}rK_1(r/l), \quad \kappa = 1 \tag{17b}$$

$$A(\zeta,x) = \frac{l^2}{\sigma\sin 2\psi}\mathrm{Re}\,[H_0^{(1)}(\xi r)], \quad 0 \leqslant \kappa < 1 \tag{17c}$$

$$B(\zeta,x) = \frac{2}{\pi\sigma}K_0(\lambda r) \tag{17d}$$

$$\sigma = 4D^3\left(\frac{1-\mu}{2}\right)^2\lambda^2\left(1+\frac{G_f}{C}\right), \tag{17e}$$

where K_0 and K_1 are the modified Bessle functions, $H_0^{(1)}$ is the Hankel function (Abramowitz and Stegun, 1965). $\kappa = \cos 2\psi$, $(1-\kappa^2)^{1/2} = \sin 2\psi$, $\psi \in [0,\pi/4]$, $r^2 = (\chi_1(x)-\chi_1(\zeta))^2+(\chi_2(x)-\chi_2(\zeta))^2$, $\xi = \mathrm{e}^{i(\pi/2+\psi)}/l$. In what follows, we consider only the case of $0 \leqslant \kappa < 1$ which seems to be valid for the most common foundation materials. The other cases are similar. Combining eqns (17), (16) and (11), and considering eqns (1) and (4), we can obtain the fundamental solutions of the generalized displacements and boundary tractions for Reissner's plate on a two parameter foundation, which are given in the Appendix.

BOUNDARY INTEGRAL EQUATIONS

The boundary integral equations for Reissner's plates on a two parameter foundation can be obtained by the method of weighted residuals. The final formulation is of the form:

$$C_{ij}(\zeta)U_j(\zeta)+\int_s P_{i\alpha}^*(\zeta,x)U_\alpha(x)\,ds+\int_s\left(P_{i3}^*(\zeta,x)+G_f\frac{\partial U_{i3}^*(\zeta,x)}{\partial n}\right)W(x)\,ds$$

$$=\int_s U_{i\alpha}^*(\zeta,x)P_\alpha(x)\,ds+\int_s U_{i3}^*(\zeta,x)\left(P_3(x)+G_f\frac{\partial W(x)}{\partial n}\right)ds+\int_\Omega U_{i3}^*(\zeta,x)q(x)\,d\Omega, \quad (18)$$

in which $C_{ij}(\zeta)=\delta_{ij}$ if $\zeta\in\Omega$, $C_{ij}(\zeta)=\delta_{ij}/2$ if ζ is on the smooth boundary, S. If ζ is on the nonsmooth boundary, S, $C_{ij}(\zeta)$ depends on the geometry of the boundary (Vander Weeën, 1982).

There are three boundary integral equations in eqn (8). The number of boundary integral equations is equal to the number of boundary unknowns for simply-supported and clamped boundaries. Thus eqn (18) can be solved. There are four boundary unknowns for a free boundary. So the fourth boundary integral equation must be established. By writing the weighted residual equation corresponding to eqn (6) and using eqn (7), the fourth boundary integral equation can be expressed as follows:

$$C_f W(\zeta)=\int_s W(x)\frac{\partial W_f^*(\zeta,x)}{\partial n}\,ds-\frac{1}{G_f}\int_s W_f^*(\zeta,x)\left(P_3(x)+G_f\frac{\partial W(x)}{\partial n}\right)ds, \quad (19)$$

where $C_f=1$ if $\zeta\in\Omega_f$, $C_f=1/2$ if ζ is on the smooth boundary, S. If ζ is on the nonsmooth boundary, C_f depends on the geometry of the boundary. The asterisked kernel functions of eqns (18) and (19) are given in the Appendix.

NUMERICAL IMPLEMENTATION

The analytical solutions of eqns (18) and (19) are not easy to obtain, and hence it is necessary to solve the boundary integral equations (18) and (19) numerically. For the numerical implementation of the present formulation, the following procedure is used. The boundary of the plate is divided into N elements with M nodes. The generalized boundary displacements (W and ψ_α) and tractions [M_n, M_{nt} and $P_3+G_f(\partial W/\partial n)$] are to be defined in terms of their nodal values. Over every boundary element between two nodes variables are interpolated linearly. In the discretization procedure, continuous elements are employed away from the corner point and partially discontinuous elements (Patterson and Sheikh, 1984) in close to the corner point to avoid the treatment of the corner point. By using the above procedure and introducing boundary conditions, eqn (18) for simply-supported and clamped boundaries, or eqns (18) and (19) for free boundary, are reduced to a system of linear algebraic equations.

Solving the resulting linear algebraic equations, all boundary displacements and tractions are known. If results are required at internal points, eqn (18) with $C_{ij}=\delta_{ij}$ can be used for calculating the displacements. Stress resultants at internal points can be calculated by coupling eqn (18) with $C_{ij}=\delta_{ij}$ and eqns (1) with the derivatives being taken with respect to the coordinates of ζ (Karam and Telles, 1988).

APPLICATIONS

Two examples are studied to verify the correctness of the present formulation and demonstrate the accuracy of the solution. For each example, the plate is subjected to a uniform distributed load q. Thirty-two boundary elements are employed on the entire boundary for each example. The results obtained are compared with the analytical solution of thin plates for the thickness h being very small.

Example 1. This example is a simply-supported thick square plate on a two parameter foundation with side length a and modulus of elastic foundation $K_f(k_f a^4/D)=200$. The

Table 1. W_c and M_c for the simply-supported thick square plate on a two parameter foundation with $K_f = 200$ and $\mu = 0.25$

	$G_F = 5$		$G_F = 20$	
h/a	$\bar{W}_cD/qa^4$	$\bar{M}_c/qa^2$	$\bar{W}_cD/qa^4$	$\bar{M}_c/qa^2$
0.005	0.2249633	2.406552	0.1559777	1.606397
0.1	0.2303082	2.352117	0.1581634	1.563309
0.2	0.2440886	2.198786	0.1637296	1.444638
Thin plate†	0.2263888	2.417870	0.1567556	1.612893

†The solutions of thin plates are obtained by the method of trigonometric series.

Table 2. W_c, M_c and M_b of the clamped thick circular plate on a two parameter foundation with $K_f = 200$ and $\mu = 0.3$

		$G_F = 5$			$G_F = 20$	
h/a	$\bar{W}_cD/qa^4$	$\bar{M}_c/qa^2$	$\bar{M}_b/qa^2$	$\bar{W}_cD/qa^4$	$\bar{M}_c/qa^2$	$\bar{M}_b/qa^2$
0.005	0.4447239	1.676206	−5.521532	0.3383348	1.162126	−5.021927
0.1	0.4467960	1.600244	−5.282508	0.3412689	1.124891	−4.708540
0.2	0.4520058	1.407280	−4.689815	0.3484621	1.028416	−3.896167
Yu, 1957	0.4448609	1.675682	−5.495585	0.3384423	1.161785	−4.994785

results of the center deflections, W_c, and center bending moments, M_c, are shown in Table 1 for the different values of $G_F(G_f a^2/D)$ and the thickness, h.

Example 2. This example is a clamped thick circular plate on two parameter foundation with radius a and modulus of elastic foundation $K_f = 200$. The results of W_c, M_c and the boundary bending moments M_c are shown in Table 2 for different values of G_F and the thickness h, and compared with the analytical solutions of thin plates on a two parameter foundation.

In Tables 1 and 2, $W_c = \bar{W}_c \times 10^{-2}$, $M_c = \bar{M}_c \times 10^{-2}$ and $M_b = \bar{M}_b \times 10^{-2}$.

CONCLUSIONS

This paper has presented the fundamental solutions and boundary integral equation formulation of Reissner's plates on two parameter foundations. On the basis of the formula presented, two numerical examples have been calculated with the application of partially discontinuous and continuous elements. From Tables 1 and 2, we see that the solutions are close to the analytical solutions of thin plates on a two parameter foundation for h being very small when using a small number of boundary elements. This fact illustrates the correctness of the present formulation and the accuracy of the solutions. The treatment of free edges in this paper is only adapted to the Pasternak type foundation.

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APPENDIX

The asterisked kernel functions which appeared in eqns (18) and (19) are as follows:

$$U^*_{\alpha\beta}(\zeta,x) = \frac{1}{\pi D(1-\mu)}[B_1(x)\delta_{\alpha\beta} - A_1(x) r_{,\alpha} r_{,\beta}] + \frac{l^2}{4D(1-\mu)\lambda^2 \sin 2\psi} \times \left\{\frac{1}{r}[\psi_3 + \alpha_2\psi_1]\delta_{\alpha\beta} + \left[\psi_4 + \alpha_2\psi_2 - \frac{2}{r}(\psi_3 + \alpha_2\psi_1)\right] r_{,\alpha} r_{,\beta}\right\}$$

$$U^*_{\alpha 3}(\zeta,x) = -U^*_{3\alpha}(\zeta,x) = \frac{l^2}{8D\alpha_3 \sin 2\psi}\psi_1 r_{,\alpha}$$

$$U^*_{33}(\zeta,x) = -\frac{l^2}{8D\alpha_3 \sin 2\psi}\left(\frac{2}{(1-\mu)\lambda^2}\psi_2 + \psi_0\right)$$

$$\frac{\partial U^*_{\alpha 3}(\zeta,x)}{\partial n} = \frac{l^2}{8D\alpha_3 \sin 2\psi}\left[\psi_2 r_{,\alpha} r_{,n} + \frac{1}{r}\psi_1(n_\alpha - 2r_{,\alpha} r_{,n})\right]$$

$$\frac{\partial U^*_{33}(\zeta,x)}{\partial n} = \frac{l^2}{8D\alpha_3 \sin 2\psi}\left[\frac{2}{(1-\mu)\lambda^2}\psi_3 + \psi_1\right] r_{,n}$$

$$W^*_f(\zeta,x) = -\frac{1}{2\pi}K_0(\lambda_1 r)$$

$$\frac{\partial W^*_f(\zeta,x)}{\partial n} = \frac{\lambda_1}{2\pi}K_1(\lambda_1 r) r_{,n}$$

$$P^*_{\gamma\alpha}(\zeta,x) = \frac{1}{2\pi r}[zK_1(z)(2r_{,\gamma} r_{,\alpha} r_{,n} - \delta_{\gamma\alpha} r_{,n} - r_{,\alpha} n_\gamma) + 2A_1(z)(4r_{,\gamma} r_{,\alpha} r_{,n} - \delta_{\gamma\alpha} r_{,n} - r_{,\alpha} n_\gamma - r_{,\gamma} n_\alpha)] + \frac{l^2}{8\lambda^2 \sin 2\psi}\left\{-(\psi_3 + \alpha_2\psi_1)\left(2r_{,\gamma} r_{,\alpha} r_{,n} + \frac{2\mu}{1-\mu} r_{,\gamma} n_\alpha\right) + \left[\frac{4}{r^2}(\psi_3 + \alpha_2\psi_1) - \frac{2}{r}(\psi_4 + \alpha\psi_2)\right](4r_{,\alpha} r_{,\gamma} r_{,n} - \delta_{\gamma\alpha} r_{,n} - r_{,\alpha} n_\gamma - r_{,\gamma} n_\alpha)\right\}$$

$$P^*_{3\alpha}(\zeta,x) = -\frac{l^2}{8\alpha_3 \sin 2\psi}\left[\psi_2((1-\mu) r_{,\alpha} r_{,n} + \mu n_\alpha) + \frac{1-\mu}{r}\psi_1(n_\alpha - 2r_{,\alpha} r_{,n})\right]$$

$$P^*_{\gamma 3}(\zeta,x) = \frac{\lambda^2}{2\pi}(B_1(z) n_\gamma - A_1(z) r_{,\gamma} r_{,n}) + \frac{l^2}{8 \sin 2\psi}\left[\frac{1}{r}(\psi_3 + \alpha_1\psi_1)(n_\gamma - 2r_{,\gamma} r_{,n}) + (\psi_4 + \alpha_1\psi_2) r_{,\gamma} r_{,n}\right]$$

$$P^*_{33}(\zeta,x) = \frac{l^2}{8\alpha_3 \sin 2\psi}\psi_3 r_{,n},$$

in which

$$\alpha_1 = \frac{k_f}{C+G_f}, \quad \alpha_2 = \alpha - \frac{1-\mu}{2}\lambda^2, \quad \alpha_3 = 1 + \frac{G_f}{C}, \quad z = \lambda r,$$

$$\lambda_1 = (k_f/G_f)^{1/2}, \quad r_{,\alpha} = \frac{\partial r}{\partial \chi_\alpha(x)}, \quad r_{,n} = r_{,\alpha} n_\alpha,$$

$$\psi_0 = 2\,\mathrm{Re}[H_0^{(1)}(\xi r)], \quad \psi_1 = 2\,\mathrm{Re}[\xi H_1^{(1)}(\xi r)],$$

$$\psi_2 = 2\,\mathrm{Re}[\xi^2 H_0^{(1)}(\xi r)], \quad \psi_3 = 2\,\mathrm{Re}[\xi^3 H_1^{(1)}(\xi r)],$$

$$\psi_4 = 2\,\mathrm{Re}[\xi^4 H_0^{(1)}(\xi r)], \quad \psi_5 = 2\,\mathrm{Re}[\xi^5 H_1^{(1)}(\xi r)],$$

$$A_1(z) = K_0(z) + \frac{2}{z}K_1(z), \quad B_1(z) = K_0(z) + \frac{1}{z}K_1(z),$$

where K_0 and K_1 are the modified Bessel functions and $H_0^{(1)}$ and $H_1^{(1)}$ are the Hankel functions (Abramowitz and Stegun, 1965).